SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Maths Class:

2022

YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 13)

- Attempt all Questions in Section II
- Allow about 1 hour and 45 minutes for this section

Examiner: AMG

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1 What is the sum of the zeroes of $Q(x) = 4x^5 + 8x^3 - 2x^2 + 5x - 2$?

A. -8

B. -2

C. 0

D. 2

2 What are the values of *x* for which $(x+3)(x-1)(2-x) > 0$?

A. $x < -3$ or $1 < x < 2$

B. $x > -3$ or $1 < x < 2$

C. $-3 < x < 1$ or $x > 2$

D. $-3 < x < 1$

3 What is the domain of $y = \frac{1}{2}\arcsin(1-5x)$?

A. $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

B. $0 \le x \le \frac{2}{5}$

C. $-\frac{2}{5} \le x \le 0$

D. $0 \le x \le \pi$

4 Which of the following expressions is a primitive of $\dfrac{1}{x^2+6x+13}$?

A. $-\dfrac{2x+6}{\left(x^2+6x+13\right)^2}$

B. $\dfrac{1}{|2x+6|}\ln\left(x^2+6x+13\right)$

C. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x+3}{2}\right)$

D. $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x+3}{2}\right)$

5 A particle is projected from the origin with initial velocity 60 m/s at an angle 30° to the horizontal.
The acceleration vector is given by

$$\ddot{\underset{\sim}{r}}(t)=\begin{pmatrix}0\\-10\end{pmatrix}, \text{ where } r(t)=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$$

is the position vector.

What are the initial conditions?

A. $\dot{x}=30\sqrt{3}, \ddot{x}=-10, \dot{y}=30$ and $\ddot{y}=0$

B. $\dot{x}=30, \ddot{x}=-10, \dot{y}=30\sqrt{3}$ and $\ddot{y}=0$

C. $\dot{x}=30, \ddot{x}=0, \dot{y}=30\sqrt{3}$ and $\ddot{y}=-10$

D. $\dot{x}=30\sqrt{3}, \ddot{x}=0, \dot{y}=30$ and $\ddot{y}=-10$

6 A curve is defined parametrically by

$$x=2\sec\theta$$
$$y=3\tan\theta$$

What is the gradient of the tangent to the curve at the point $\left(2\sec\theta, 3\tan\theta\right)$?

A. $\dfrac{3}{2}\operatorname{cosec}\theta$

B. $\dfrac{2}{3}\operatorname{cosec}\theta$

C. $\dfrac{3}{2}\sin\theta$

D. $\dfrac{2}{3}\sin\theta$

7 A bag contains 2 red, 5 blue, 6 white, 11 green and 14 yellow marbles.

What is the minimum number of marbles that need to be chosen randomly from the bag to ensure that 6 marbles of the same colour have been chosen?

A. 16

B. 17

C. 23

D. 34

8 Six identical chairs are equally spaced around a circular table.

What is the number of ways that three men and three women can be seated at the table so that no two men are opposite each other?

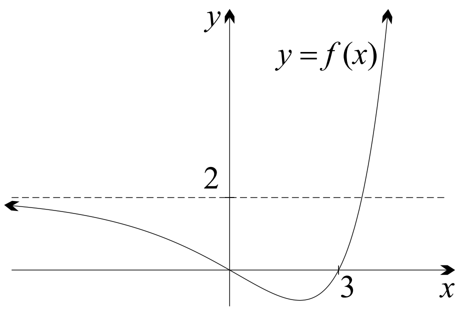
A. 12

B. 48

C. 72

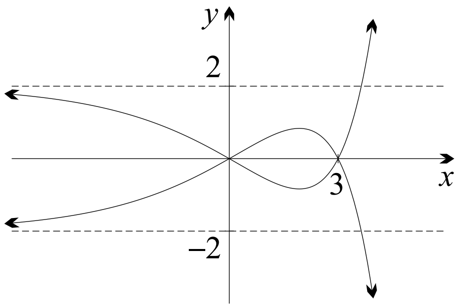
D. 288

9 The graph of $y = f(x)$ is drawn below.

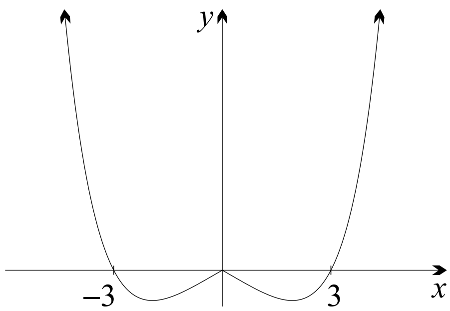


Which of the following is the best graph for $|y| = f(|x|)$?

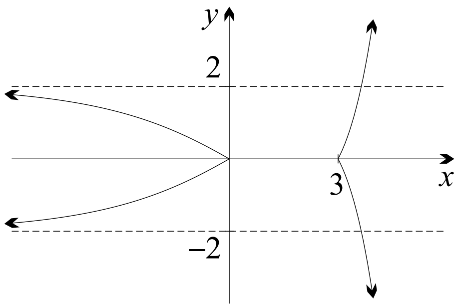
A.



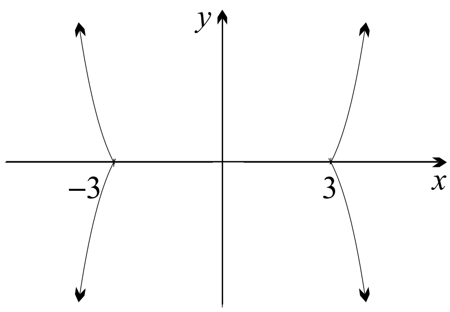
B.



C.



D.



10 Which of the following is the projection of the force $\underset{\sim}{F} = a\underset{\sim}{i} + b\underset{\sim}{j}$, where a and b are real constants, in the direction of $\underset{\sim}{w} = \underset{\sim}{i} + \underset{\sim}{j}$?

A. $\dfrac{\underset{\sim}{F}}{a+b}$

B. $\left(\dfrac{a+b}{a^2+b^2}\right)\underset{\sim}{F}$

C. $\left(\dfrac{a+b}{2}\right)\underset{\sim}{w}$

D. $\left(\dfrac{a+b}{\sqrt{2}}\right)\underset{\sim}{w}$

Section II

60 marks
Attempt Questions 11–14
Allow 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE Writing Booklet.

(a) It is given that $\cos\theta = \frac{3}{5}$ for $\frac{3\pi}{2} \le \theta \le 2\pi$.

(i) Show that $\sin 2\theta = -\frac{24}{25}$. **2**

(ii) By considering $3\theta = \theta + 2\theta$, show that $\cos 3\theta = -\frac{117}{125}$. **2**

(b) Vector $\overrightarrow{AB}$ has a magnitude of $2\sqrt{3}$ and makes an angle of 150° with the positive y-axis, measuring anticlockwise from the positive y-axis. **2**

Write a unit vector in the direction of $\overrightarrow{AB}$ in the form $a\underset{\sim}{i} + b\underset{\sim}{j}$, where a and b are constants written in simplest exact form.

(c) By considering the graphs of $y = |x-2|$ and $y = 3(x+4)$, or otherwise, solve **3**

$$\frac{|x-2|}{x+4} \le 3.$$

Question 11 continues on page 7

Question 11 (continued)

(d) When a polynomial, $p(x)$, with rational coefficients is divided by $(x-\alpha)^2$ the remainder is $R_2(x-\alpha)+R_1$, where $R_1, R_2 \in \sim$

$$\text{i.e. } p(x)=(x-\alpha)^2 Q(x)+R_2(x-\alpha)+R_1,$$

where $Q(x)$ is a polynomial.

(i) Show that $R_1 = p(\alpha)$ and $R_2 = p'(\alpha)$. **2**

(ii) For $n > 1,\ n \in \mathbb{Z}$, show that $x^n - n(x-1)-1$ is divisible by $(x-1)^2$. **2**

(iii) For $n > 1,\ n \in \mathbb{Z}$, deduce that $2^{4n} - 15n - 1$ is divisible by 225. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE Writing Booklet.

(a) A liquid with an initial temperature of 250° C is enclosed in a metal container that is kept at a constant temperature of 120° C.
The temperature of the liquid, T, after t hours satisfies the differential equation

$$\frac{dT}{dt} = k\left(T - 120\right).$$

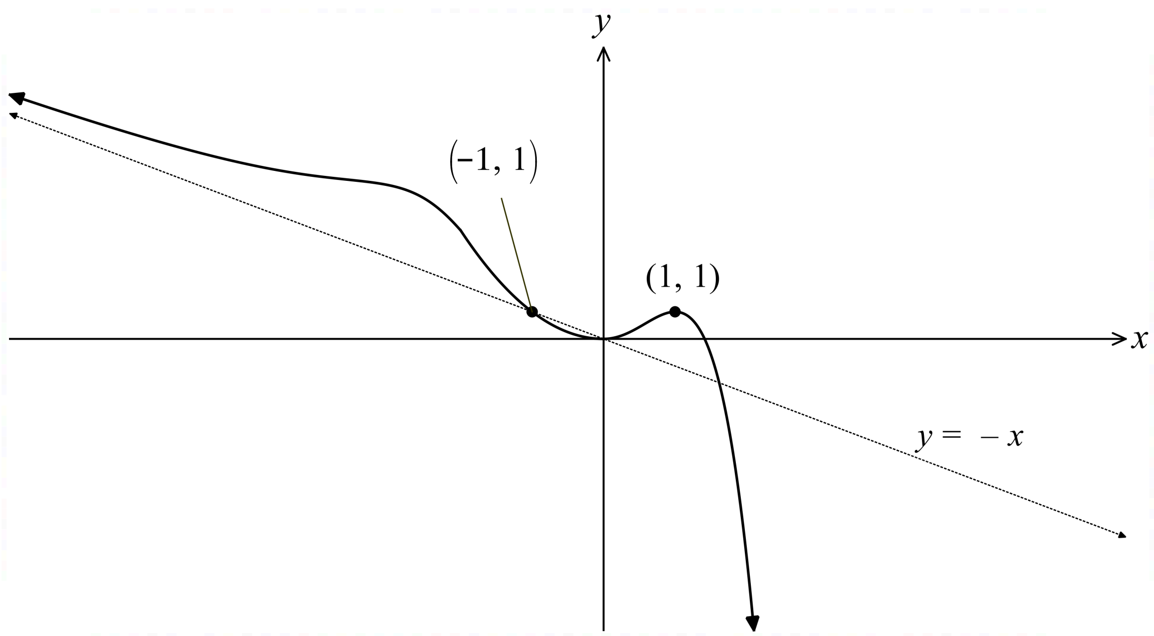
It is known that the liquid cools to 180° C in half an hour.

(i) Show that $T = 120 + Ae^{kt}$, where $k, A \in \mathbb{R}$ is a solution of the above equation. **1**

(ii) How long will it take for the liquid to reach 150°? **2**
Write your answer to the nearest hour.

(b) By letting $x + 5 = x + 6 - 1$, or otherwise, find $\displaystyle\int \frac{x+5}{\sqrt{x+6}}\, dx$. **2**

(c) The diagram shows the graph of $y = f(x)$. The line $y = -x$ is an asymptote.
On the sheet provided, sketch the graphs of the following, showing all relevant information.



(i) $\left| y \right| = f(x)$ **2**

(ii) $y = f(x) + x$ **2**

Question 12 continues on page 9

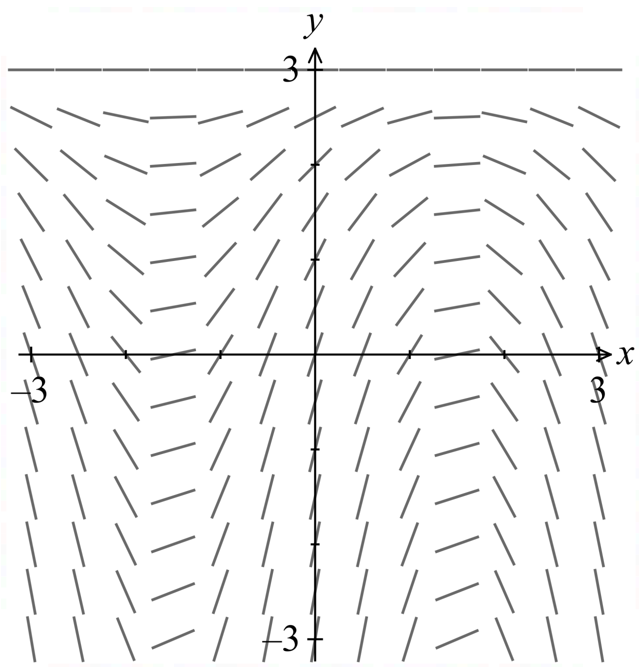
Question 12 (continued)

(d) Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. **2**

Let $y = f(x)$ be the particular solution to the differential equation with the condition $f(0) = 1$. The function f is defined for all real numbers.

A portion of the slope field of the differential equation is given below.

On the sheet provided, sketch the solution curve through the point $(0, 1)$.



(e) (i) Show that $y = \frac{1}{2x^2} \int f(x)\, dx$ is a solution of the differential equation **2**

$$2x^2 \frac{dy}{dx} + 4xy = f(x)$$

(ii) Hence, find an expression for y in terms of x that is a solution of **2**

$$2x^2 \frac{dy}{dx} + 4xy = \frac{1}{x^2 + 1} \text{ for } x > 0.$$

It is also given that $y = \frac{\pi}{4}$ when $x = 1$.

End of Question 12

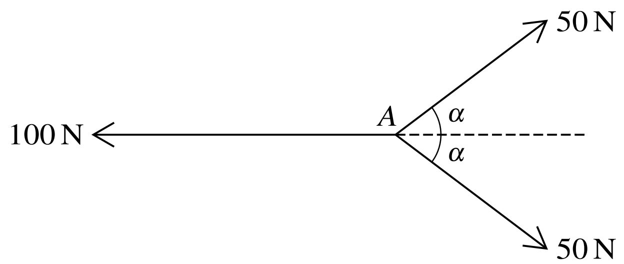
Question 13 (16 marks) Use a SEPARATE Writing Booklet.

(a) A fireworks technician is testing fireworks for an upcoming display. **2**
He has 20 fireworks, of which 13 are red and 7 are yellow.
He launches 8 of them in a random order.

What is the probability that 5 of the launched fireworks are red?

(b) Three coplanar forces of magnitude 100 N, 50 N and 50 N act at a point A, as shown in the diagram below. **3**

The angle α is such that $\cos\alpha = \frac{4}{5}$.



Find the magnitude of the resultant of the three forces and state its direction.

(c) (i) Prove, by mathematical induction that **3**

$$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$$

where $\sin\theta \neq 0$ and $n \in \mathbb{Z}^+$.

(ii) Using part (i) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise, show that **2**

$$\sin x - \sin 3x + \sin 5x = \frac{\sin 6x}{2\cos x}.$$

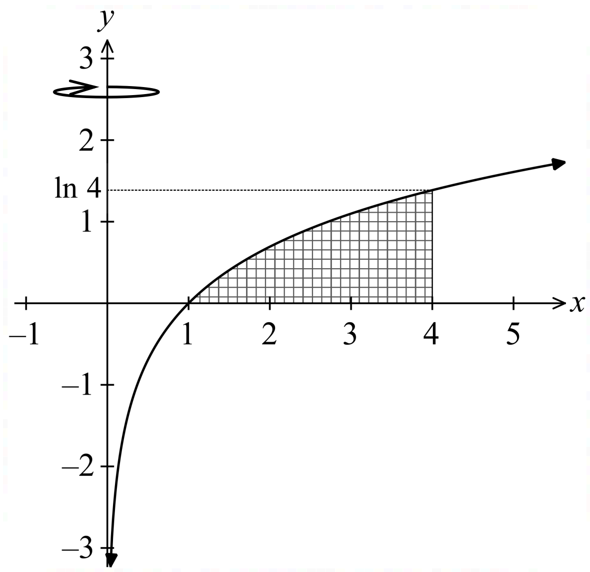
(iii) Show that $\displaystyle\int_0^{\frac{\pi}{3}} \left(\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x}\right)^2 dx = \sqrt{3} - \frac{\pi}{3}$. **3**

Question 13 continues on page 11

Question 13 (continued)

(d) The diagram below shows the graph of $y = \ln x$. **3**

The shaded region, bounded by $y = \ln x$, the line $x = 4$ and the x-axis, is rotated 360° about the y-axis to form a solid.



Find the volume of the solid, leaving it in simplest exact form.

End of Question 13

Question 14 (15 marks) Use a SEPARATE Writing Booklet.

(a) A golfer hits a golf ball from a point O with speed V m/s at an angle of 30° above the horizontal and moves freely under gravity. **4**
The ball reaches its greatest height at time T seconds after projection.

The position vector, $\underset{\sim}{r}$, at any time, t seconds, is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} \frac{\sqrt{3}}{2}Vt \\ \frac{1}{2}Vt - 5t^2 \end{pmatrix} \qquad \text{(Do \textbf{NOT} prove)}$$

Find, in terms of V, the speed of the ball at time $\dfrac{2}{3}T$ seconds after projection.

Question 14 continues on page 13

Question 14 (continued)

(b) Let L be the straight line passing through the point $P\left(x_0, y_0\right)$ with an angle of inclination θ to the x-axis.

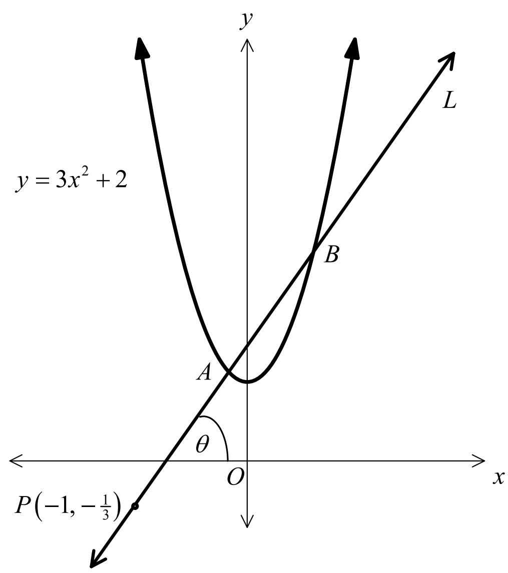
The line L is defined parametrically by

$$x = x_0 + t\cos\theta$$
$$y = y_0 + t\sin\theta$$

where $t \in \mathbb{R}$ and $\theta \in \left[0, \pi\right]$.

(i) Show that for any point T on the line L that $PT = \left| t \right|$. **2**

(ii) In the diagram below, L cuts the parabola $y = 3x^2 + 2$ at points A and B, where P is the point $\left(-1, -\frac{1}{3}\right)$. Let $PA = t_1$ and $PB = t_2$.



(α) Show that t_1 and t_2 are the roots of the equation **2**

$$9t^2\cos^2\theta - 3t\left(\sin\theta + 6\cos\theta\right) + 16 = 0.$$

(β) Hence show that $AB^2 = \dfrac{\sin^2\theta + 12\sin\theta\cos\theta - 28\cos^2\theta}{9\cos^4\theta}$ **3**

(γ) If L is a tangent show that the two possible gradients of L are 2 and –14. **2**

(δ) Hence, if one of the tangents has a gradient of 2, show that $PZ = \dfrac{4\sqrt{5}}{3}$, **2**

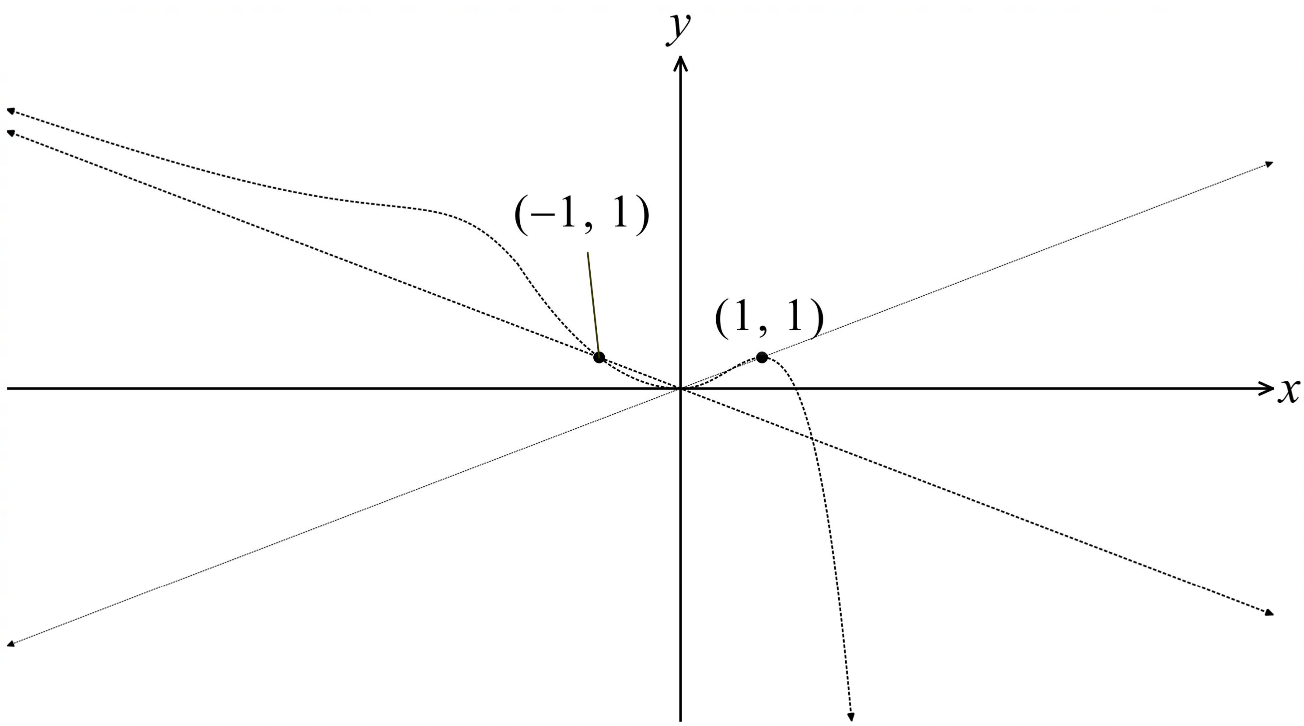
where Z is the point of tangency of the line L to the parabola.

End of paper

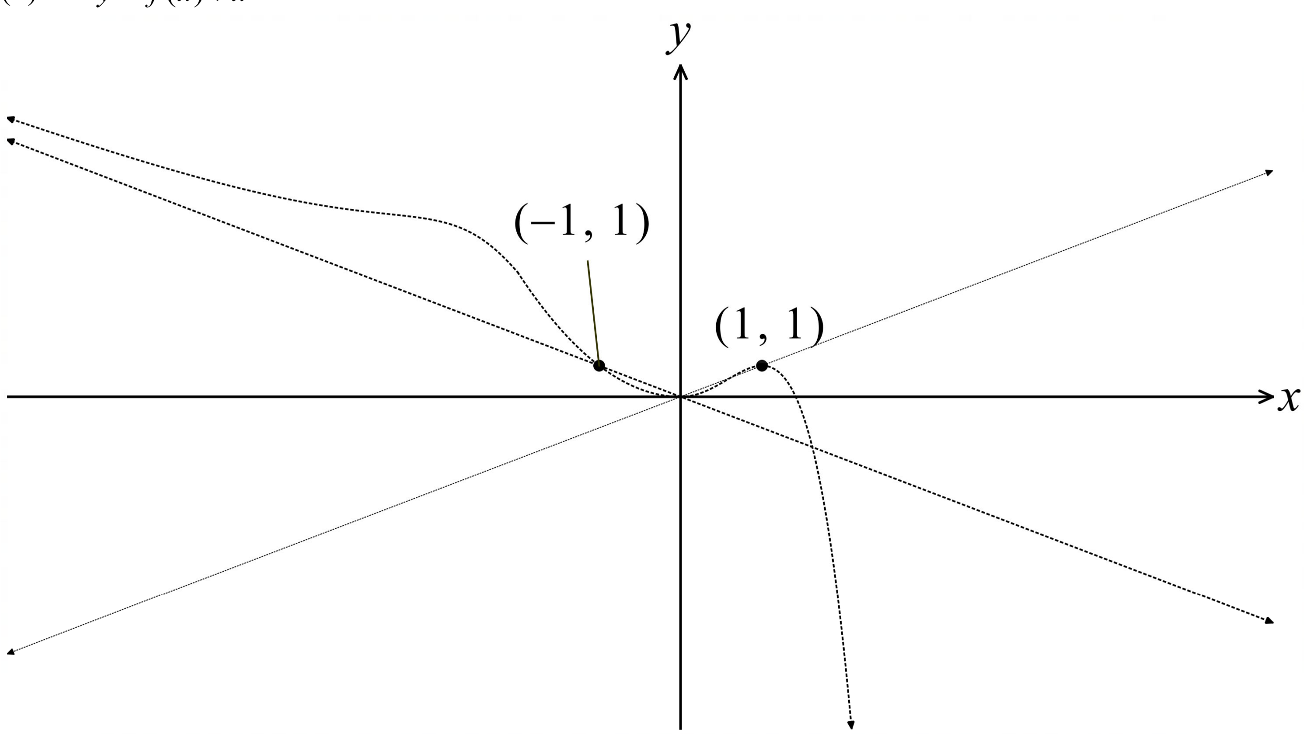
Question 12 (c) **Place inside booklet for Q12 at the end of exam**

The diagrams shows the graph of $y = f(x)$. The line $y = -x$ is an asymptote.
The graph of $y = x$ has been drawn as an aide.
Sketch the graphs of the following, showing all relevant information.

(i) $|y| = f(x)$ **2**



(ii) $y = f(x) + x$ **2**



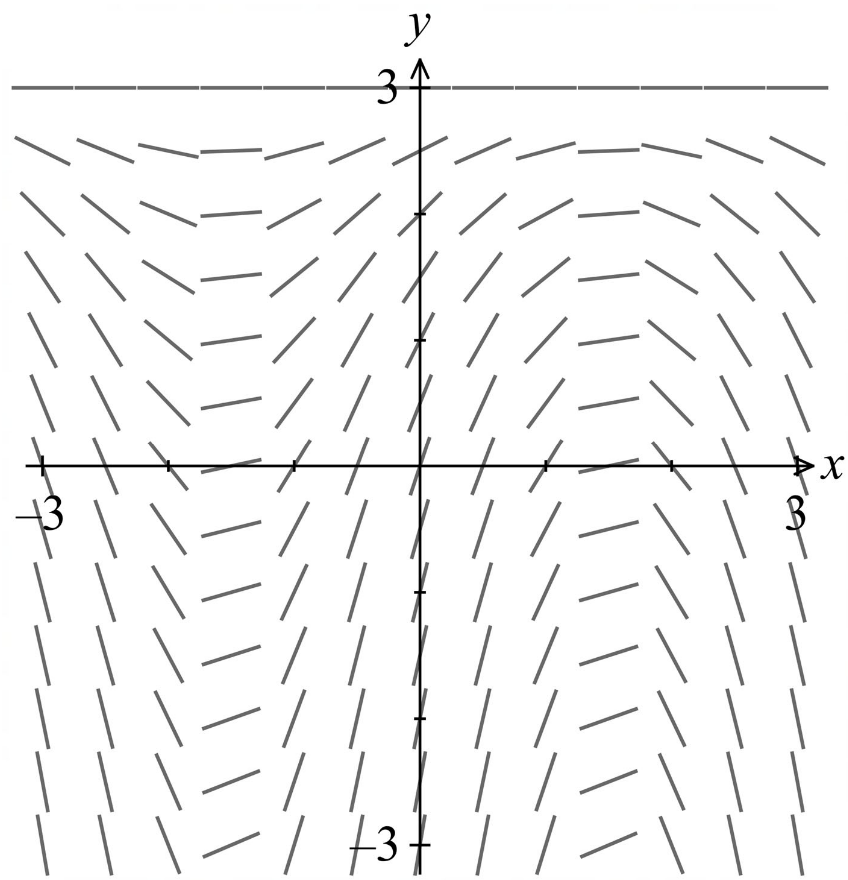
Question 12 (d) **Place inside booklet for Q12 at the end of exam**

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. **2**

Let $y = f(x)$ be the particular solution to the differential equation with the condition $f(0) = 1$.
The function f is defined for all real numbers.

A portion of the slope field of the differential equation is given below.

Sketch the solution curve through the point $(0, 1)$.



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YEAR 12
TASK 4
TRIAL HSC

Mathematics Extension 1

Sample Solutions

NOTE: Some of you may be disappointed with your mark.

This process of checking your mark is NOT the opportunity to improve your marks.
Improvement will come through further revision and practice, as well as reading the solutions and comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown ‘working’ does not justify that your solution is worth any marks.

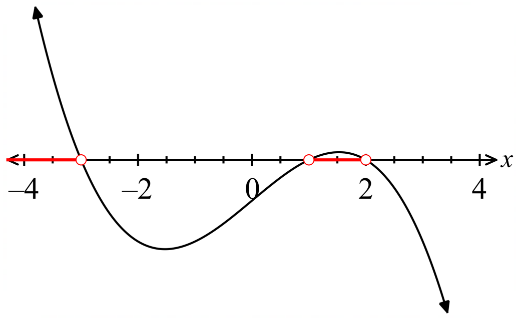
MC Answers

1	C	**6**	A
2	A	**7**	C
3	B	**8**	B
4	C	**9**	D
5	D	**10**	C

Section I Multiple Choice

1 **C** Sum of zeroes $= -\frac{0}{4} = 0$

2 **A** $x < -3$ or $1 < x < 2$



3 **B** $0 \le x \le \frac{2}{5}$

Note: Options A and D are out.

Need $-1 \le 1 - 5x \le 1$ (**Note** this is the same as $-1 \le 5x - 1 \le 1$)

$\therefore -2 \le -5x \le 0$

$\therefore 0 \le x \le \frac{2}{5}$

4 **C** $\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)$

Note: Options A and B are out.

As $x^2 + 6x + 13 = (x+3)^2 + 4$ the answer is C

$$\int \frac{1}{x^2 + 6x + 13}\,dx = \frac{1}{2}\int \frac{2}{(x+3)^2 + 2^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

5 **D** $\dot{x} = 30\sqrt{3}, \ddot{x} = 0, \dot{y} = 30$ and $\ddot{y} = -10$

Note: Options A and B are out when you consider $\ddot{y}$.

The initial velocity vector is $\begin{pmatrix} 60\cos 30° \\ 60\sin 30° \end{pmatrix}$

6 **A** $\frac{3}{2}\text{cosec}\,\theta$

Note: Options B and D are out when you consider the ratio i.e. $\frac{2}{3}$.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 3\sec^2\theta \times \frac{1}{2\sec\theta\tan\theta}$$

$$= \frac{3}{2}\sec\theta \times \cot\theta$$

$$= \frac{3}{2}\text{cosec}\theta$$

7 **C** 23

To get 5 marbles of the same colour will require 2 R, 5 B, 5 W, 5 G and 5 Y i.e. 22 balls

To ensure 6 marbles of one colour is 22 + 1 = 23 marbles

8 **B** 48 ways

Sit a man down first.

Then there are only 4 seats the second man can sit in, to avoid being opposite the first man.

So, the last man has only 2 choices.

The women can be sat down in 3! ways.

A total of $4 \times 2 \times 3! = 48$ ways.

9 **D** This is the ‘best’ graph as it is the only one with symmetry in both coordinate axes.

10 **C**

Note: Options A and B are out as they are in the direction of **F**.

$$\widehat{\mathbf{w}} = \frac{1}{\sqrt{2}}\left(\underset{\sim}{\mathrm{i}} + \underset{\sim}{\mathrm{j}}\right)$$

$$\text{proj}_{\mathbf{w}}\mathbf{F} = (\mathbf{F} \cdot \widehat{\mathbf{w}})\ \widehat{\mathbf{w}}$$

$$= \left(\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \frac{1}{\sqrt{2}}\left(\underset{\sim}{\mathrm{i}} + \underset{\sim}{\mathrm{j}}\right)\right)\frac{1}{\sqrt{2}}\left(\underset{\sim}{\mathrm{i}} + \underset{\sim}{\mathrm{j}}\right)$$

$$= \frac{1}{2}(a + b)\left(\underset{\sim}{\mathrm{i}} + \underset{\sim}{\mathrm{j}}\right)$$

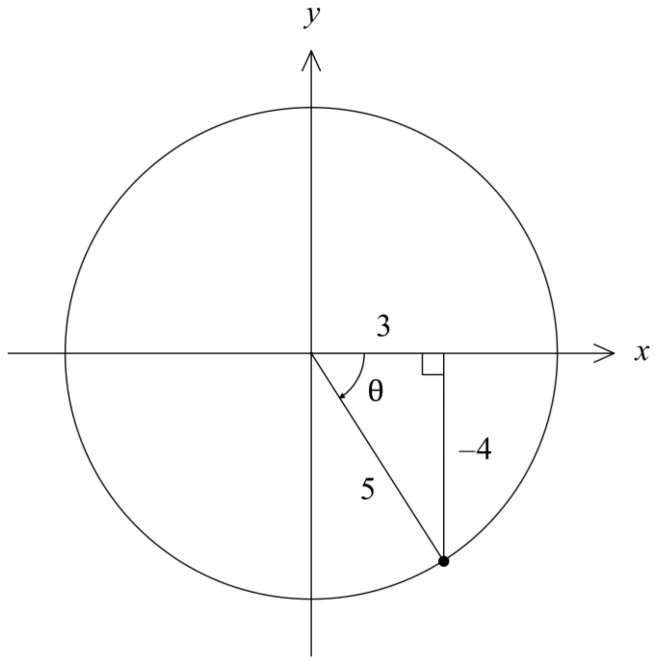
$$= \frac{1}{2}(a + b)\underset{\sim}{w}$$

2022 HSC ME1 Task 4: Q11 Solutions and Comments

Answers in pencil or with white out/liquid paper **CAN'T** appeal for marks.

A. It is given that $\cos\theta = \frac{3}{5}$ for $\frac{3\pi}{2} \le \theta \le 2\theta$.

I. Show that $\sin 2\theta = \frac{-24}{25}$. 2

Solution	Comment(s)
$\frac{3\pi}{2} \le \theta \le 2\theta$, so $\sin\theta < 0$ in the fourth quadrant.  By Pythag and right angle trig, $\sin\theta = \frac{-4}{5}$. $\therefore \sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{-4}{5} \times \frac{3}{5} = \frac{-24}{25}$	Students who found $\sin 2\theta$ by calculating $\theta = 53°$ couldn't score any marks, even if their θ value was stored in memory, as calculator values are always approximations. Students who assumed that 2θ is in the fourth quadrant because θ is in the fourth quadrant risked being penalised for the assumption.

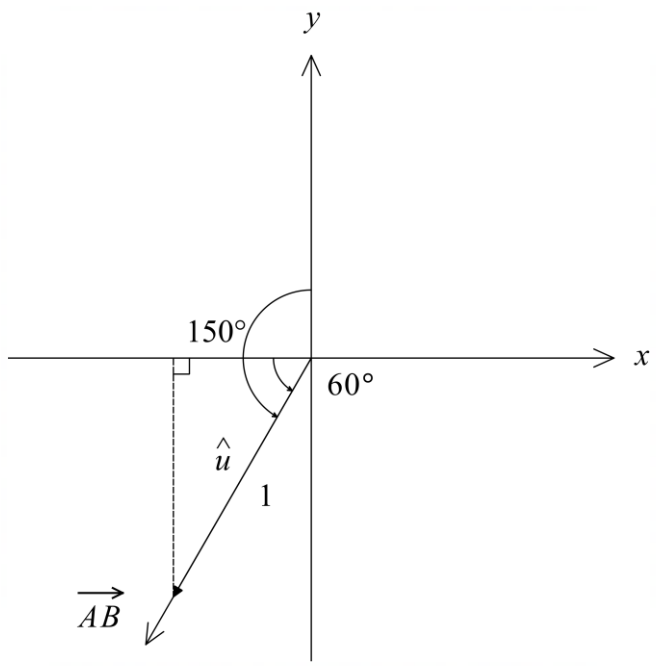
II. By considering $3\theta = \theta + 2\theta$, show that $\cos 3\theta = \frac{-117}{125}$. 2

Solution	Comment(s)
$\cos 3\theta = \cos(\theta + 2\theta)$ $= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$ $= \cos\theta\,(\cos^2\theta - \sin^2\theta) - \sin\theta\sin 2\theta$ $= \frac{3}{5}\left(\left(\frac{3}{5}\right)^2 - \left(\frac{-4}{5}\right)^2\right) - \left(\frac{-4}{5} \times \frac{-24}{25}\right)$ $= \frac{-117}{125}$	Some students were penalised for failing to explicitly show why $\cos\theta = \frac{-7}{25}$.

Answers in pencil or with white out/liquid paper **CAN'T** appeal for marks.

B. Vector $\overrightarrow{AB}$ has a magnitude of $2\sqrt{3}$ and makes an angle of $150°$ with the positive y axis, measuring anticlockwise from the positive y axis. **2**

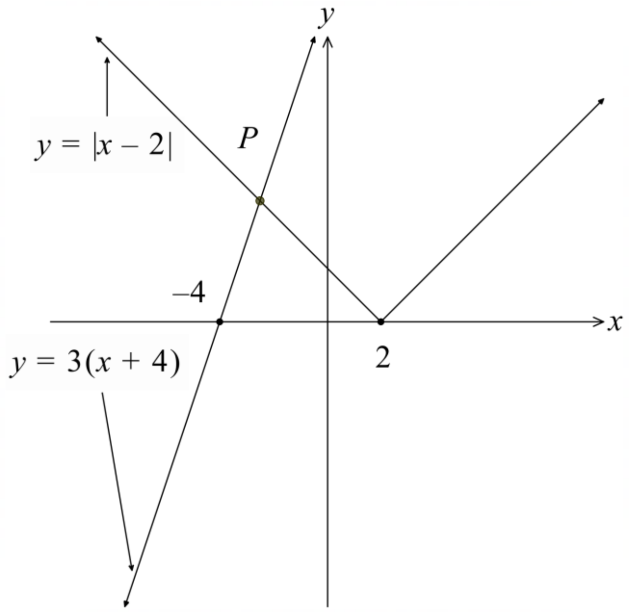
Write a unit vector in the direction of $\overrightarrow{AB}$ in the form $a\underset{\sim}{i} + b\underset{\sim}{j}$, where a and b are constants written in simplest exact form.

Solution	Comment(s)
Let $\hat{u}$ be the unit vector of $\overrightarrow{AB}$. Then, $\hat{u}$ points in the same direction as $\overrightarrow{AB}$ and has a magnitude of 1 unit.  $150°$ anticlockwise from the positive y axis is $150° - 90° = 60°$ in the third quadrant, where $\cos\theta < 0$ and $\sin\theta < 0$. Hence, by right angle trig: $\hat{u} = (-\cos 60°)\hat{\imath} + (-\sin 60°)\hat{\jmath}$ $= \frac{-1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{\jmath}$	Many students failed to read the question carefully, as indicated by the following common errors: • Finding the vector $\overrightarrow{AB}$ rather than a unit vector of $\overrightarrow{AB}$. • Using the positive x axis rather than the positive y axis as a reference for the angle.

<u>2022 HSC ME1 Task 4: Q11 Solutions and Comments</u>

Answers in pencil or with white out/liquid paper **CAN'T** appeal for marks.

C. By considering the graphs of $y = |x - 2|$ and $y = 3(x + 4)$, or otherwise, solve $\dfrac{|x-2|}{x+4} \leq 3$. 3

<table>
<tr><th>Solution</th><th>Comment(s)</th></tr>
<tr>
<td>
The graphs of $y = |x - 2|$ and $y = 3(x + 4)$ are shown below:


If $x < -4$, then $|x - 2| > 0$ and $3(x + 4) < 0$. A positive expression divided by a negative expression will always give a negative expression, which is clearly always less than or equal to 3. Hence, $x < -4$ is a solution.

If $x > -4$, then $|x - 2| > 0$ and $3(x + 4) > 0$. Hence, if $\dfrac{|x-2|}{x+4} \leq 3$, then $|x - 2| \leq 3(x + 4)$, which occurs at P.

Solving for the x coordinate of P:
$$-(x-2) = 3(x+4)$$
$$-x + 2 = 3x + 12$$
$$4x = -10$$
$$x = \frac{-5}{2}$$
Hence, $\dfrac{|x-2|}{x+4} \leq 3$ when $x < -4$ or $x \geq \dfrac{-5}{2}$.
</td>
<td>
Alternatively, from the graph, the critical points to examine are $x = -4, \dfrac{-5}{2}, 2$.

Hence, substituting $x = 3$ will show that the inequality holds true for $x \geq 2$.

Similarly, the inequality:
• Holds true for $\dfrac{-5}{2} \leq x \leq 2$.
• Doesn't hold true for $-4 < x \leq \dfrac{-5}{2}$.
• Holds true for $x < -4$

Hence, the solution is $x < -4$ or $x \geq \dfrac{-5}{2}$, as before.

Common error(s):
• Failing to address the case of $x < -4$.
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2022 HSC ME1 Task 4: Q11 Solutions and Comments

Answers in pencil or with white out/liquid paper **CAN'T** appeal for marks.

D. When a polynomial $p(x)$ with rational coefficients is divided by $(x-\alpha)^2$, the remainder is $R_2(x-\alpha)+R_1$, where $R_1, R_2 \in \mathbb{R}$,

$$\text{i.e. } p(x) = (x-\alpha)^2 Q(x) + R_2(x-\alpha) + R_1,$$

where $Q(x)$ is a polynomial.

I. Show that $R_1 = p(\alpha)$ and $R_2 = p'(\alpha)$. 2

Solution	Comment(s)
Subbing in $x=\alpha$ into $p(x)$: $p(\alpha) = (\alpha-\alpha)^2 Q(\alpha) + R_2(\alpha-\alpha) + R_1$ $= R_1$ Differentiating $p(x)$ wrt x using the product rule: $p'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x) + R_2$ Subbing in $x=\alpha$ into $p'(x)$: $p'(\alpha) = 2(\alpha-\alpha)Q(\alpha) + (\alpha-\alpha)^2 Q'(\alpha) + R_2$ $= R_2$	Common error(s): • Incorrectly differentiating $p(x)$.

II. For $n > 1, n \in \mathbb{Z}$, show that $x^n - n(x-1) - 1$ is divisible by $(x-1)^2$. 2

Solution	Comment(s)
Let $p(x) = x^n - n(x-1) - 1$. As $p(x)$ has rational coefficients, from Part I, $p(x)$ divided by $(x-1)^2$ will have a remainder $R_2(x-1)+R_1$, where $R_1 = p(1)$ and $R_2 = p'(1)$. $p(1) = 1^n - n(1-1) - 1$ $= 0$ $p'(x) = nx^{n-1} - n$ $= n(x^{n-1} - 1)$ $p'(1) = n(1^{n-1} - 1)$ $= 0$ $R_2(x-1) + R_1 = 0(x-1) + 0$ $= 0$ Hence, $x^n - n(x-1) - 1$ is divisible by $(x-1)^2$.	Students would be wise to not consider induction as a viable alternative, as the process is long, inefficient and prone to mistakes. Students who assumed that the divisibility holds true for $n > 1$ because they showed that it holds true for $n = 2$ couldn't score any marks, as they've failed to understand the process of proof by induction.

III. For $n > 1, n \in \mathbb{Z}$, deduce that $2^{4n} - 15n - 1$ is divisible by 225. 1

Solution	Comment(s)
$2^{4n} - 15n - 1 = (2^4)^n - 15n - 1$ $= 16^n - n(16-1) - 1$ From Part II, $2^{4n} - 15n - 1$ is divisible by $(16-1)^2 = 225$.	Again, students shouldn't consider induction as an alternative, especially given Part II above. Also, students who proved the divisibility to be true for $n = 2$ couldn't score any marks, as before.

Q12 (a)

$$\frac{dT}{dt} = k(T - 120)$$

(i) $T = 120 + Ae^{kt}$ (1)

Show that (1) is a solution

$$\frac{dT}{dt} = kAe^{kt}$$

but $Ae^{kt} = T - 120$ (from (1))

$$\Rightarrow \frac{dT}{dt} = k(T - 120) \therefore (1) \text{ is a solution}$$

[1 mark] Students used integration as well to show this was a solution.

(ii) $T = 120 + Ae^{kt}$

When $t = 0, T = 250$

$$\Rightarrow 250 = 120 + Ae^0$$

$$\therefore A = 130$$

$$\Rightarrow T = 120 + 130e^{kt}$$

When $t = 0.5, T = 180$

$$\Rightarrow 180 = 120 + 130e^{0.5k}$$

$$60 = 130e^{0.5k}$$

$$\frac{6}{13} = 130e^{0.5k}$$

$$\ln\frac{6}{13} = 0.5k$$

$$\Rightarrow k = 2\ln\frac{6}{13} \qquad (\approx -1.546)$$

$$\Rightarrow T = 120 + 130e^{\left(2\ln\frac{6}{13}\, t\right)}$$

Find t when $T = 150$

$$\Rightarrow 150 = 120 + 130e^{(2\ln\frac{6}{13}\, t)}$$

$$\ln\frac{3}{13} = 2\ln\frac{6}{13}t$$

$$t = \ln\frac{3}{13} \div 2\ln\frac{6}{13}$$

$$t \approx 0.948 \text{ h}$$

$$t \approx 1 \text{ h}$$

[2 marks] This question was also done very well by students

Q12 (b)

$$\int \frac{x+5}{\sqrt{x+6}}\,dx$$

$$= \int \frac{x+6-1}{\sqrt{x+6}}\,dx$$

$$= \int \sqrt{x+6} - (x+6)^{-1/2}\,dx$$

$$= \frac{2}{3}(x+6)^{\frac{3}{2}} - 2(x+6)^{\frac{1}{2}} + C$$

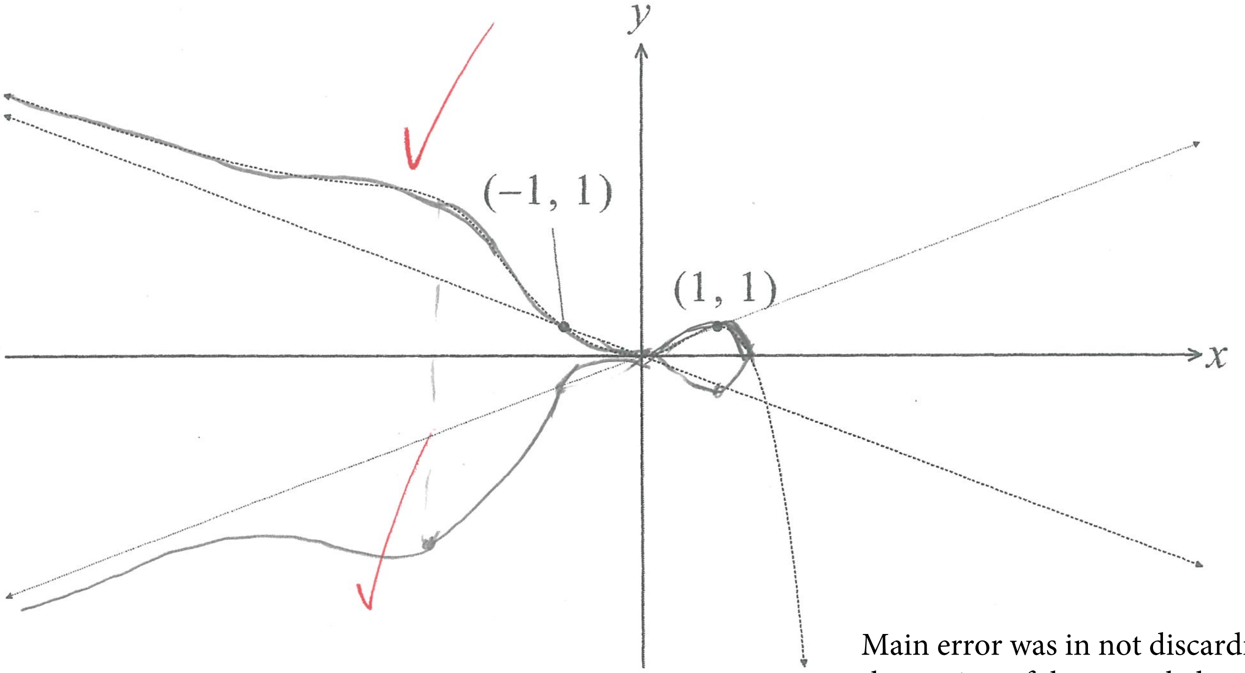
$$= \frac{2}{3}(x+6)^{\frac{3}{2}} - 2\sqrt{x+6} + C$$

[2 marks] Most students did this well. Some used a 'u' substitution successfully too. Some errors resulted from using index rules incorrectly.

Question 12 (c) **Place inside booklet for Q12 at the end of exam**

The diagrams shows the graph of $y = f(x)$. The line $y = -x$ is an asymptote.
The graph of $y = x$ has been drawn as an aide.
Sketch the graphs of the following, showing all relevant information.

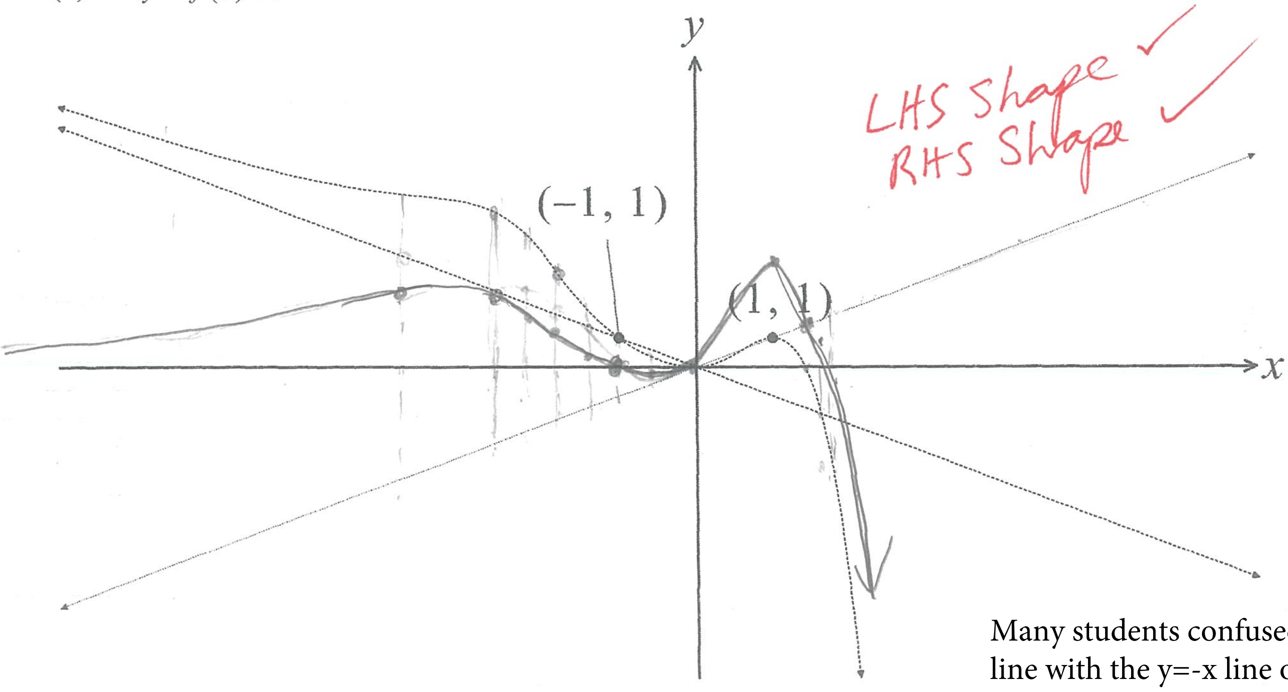
(i) $|y| = f(x)$ **2**



Main error was in not discarding the section of the curve below the x axis as the first step.

(ii) $y = f(x) + x$ **2**

LHS Shape ✓
RHS Shape ✓



Many students confused the y-=x line with the y=-x line on the left hand side.

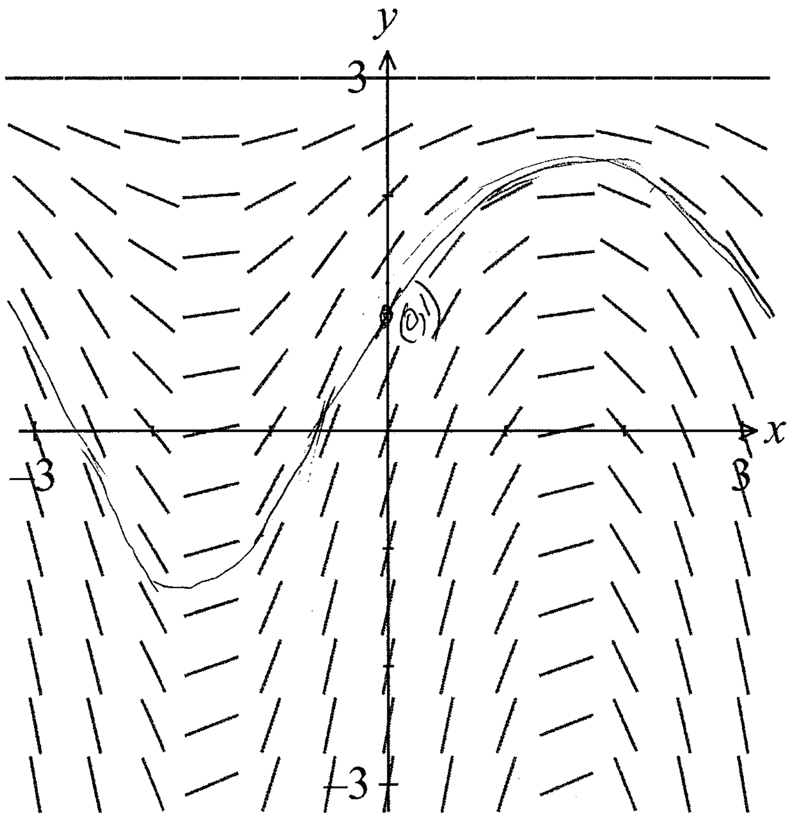
Question 12 (d) **Place inside booklet for Q12 at the end of exam**

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. **2**

Let $y = f(x)$ be the particular solution to the differential equation with the condition $f(0) = 1$.
The function f is defined for all real numbers.

A portion of the slope field of the differential equation is given below.

Sketch the solution curve through the point $(0, 1)$.



Shape ✓
In channel ✓

Many students crossed the channel and lost a mark.

Q12 (e) *(i)*

$$y = \frac{1}{2x^2}\int f(x)\,dx \qquad (2)$$

$$\Rightarrow 2x^2 y = \int f(x)\,dx$$

Differentiate both sides w.r.t. x

$$\Rightarrow 2x^2\frac{dy}{dx} + 4xy = f(x)$$

$\therefore$ (2) is a solution

[2 marks] One error was in starting with this last line and then trying to integrate terms which contained products

(ii)

$$y = \frac{1}{2x^2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2x^2}(\tan^{-1} x + C)$$

When $x = 1, y = \frac{\pi}{4}$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2}\left(\tan^{-1} 1 + C\right)$$

$$\frac{\pi}{2} = \frac{\pi}{4} + C$$

$$C = \frac{\pi}{4}$$

$$\Rightarrow y = \frac{1}{2x^2}\left(\tan^{-1} x + \frac{\pi}{4}\right)$$

[2 marks] Only about a third of students who recognised the inverse tan function proceeded correctly from there. The main error was in not having the constant of integration in brackets. This resulted in an incorrect value of C.

Q13 (a)

Sample space size $= {}^{20}C_8 = \dfrac{20!}{8! \times 12!}$

Ways to get 5 red $= {}^{13}C_5 = \dfrac{13!}{5! \times 8!}$

Ways to get 3 yellow $= {}^{7}C_3 = \dfrac{7!}{3! \times 4!}$

$$\text{So } P = \frac{{}^{13}C_5 \times {}^{7}C_3}{{}^{20}C_8} = \frac{13! \times 7! \times \cancel{8!} \times 12!}{20! \times 5! \times \cancel{8!} \times 3! \times 4!} = \frac{231}{646}$$

Notes:

1. It's not primary school, so don't express probability as a percentage
2. Note that there is no replacement so don't use a binary distribution in which the probabilities at each stage are constant

Q13 (b)

Vertical force component $= 50N(\sin\alpha + \sin(-\alpha)) = 0N$

Horizontal force component $= 50N(\cos\alpha + \cos(-\alpha)) - 100N = \frac{8}{5} \times 50N - 100N = -20N$

$\therefore$ The resultant force is $20N$ to the left.

Notes:

1. Take a common sense guess first to see if your answer matches it
2. No need to use compass directions
3. The trig result is given, so no need to calculate it
4. Mostly done well.

Q13 (c) *(i)*

<u>For $n = 1$:</u> $(2n - 1 = 1)$

$$\text{RHS} = \frac{\sin 2\theta}{2\sin\theta} = \frac{\cancel{2\sin\theta}\cos\theta}{\cancel{2\sin\theta}} = \cos\theta = \text{LHS}.$$

So true for $n = 1$.

Suppose it is true for $n = k$ i.e. $\displaystyle\sum_{i-1}^{k}(\cos(2i-1)\theta) = \frac{\sin 2k\theta}{2\sin\theta}$

<u>For $n = k + 1$:</u>

$$\begin{aligned}
\sum_{i=1}^{k+1}(\cos(2i-1)\theta) &= \frac{\sin 2k\theta}{2\sin\theta} + \cos((2k+1)\theta) && \text{(from assumption)} \\
&= \frac{\sin 2k\theta}{2\sin\theta} + \frac{2\sin\theta\cos(2k\theta+\theta)}{2\sin\theta} \\
&= \frac{\cancel{\sin 2k\theta} - \cancel{\sin 2k\theta} + \sin(2k\theta + 2\theta)}{2\sin\theta} && \left(\text{since } \sin A\cos B = \frac{1}{2}(\sin(A+B) - \sin(A-B))\right) \\
&= \frac{\sin 2(k+1)\theta}{2\sin\theta}
\end{aligned}$$

$\therefore$ true for $n = k + 1$ and for all n by principle of mathematical induction $\square$

Note: Don't forget the trig formula used here. There are other ways though, but slighter longer.

(c) (ii) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$

$\cos(x \pm 2\pi) = \cos x$ and $\sin(x \pm 2\pi) = \sin x$

$$\sin x - \sin 3x + \sin 5x = \sin\left(\frac{\pi}{2} - \theta\right) - \sin\left[3\left(\frac{\pi}{2} - \theta\right)\right] + \sin\left[5\left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \cos\theta - \sin\left[\frac{\pi}{2} - (3\theta - \pi)\right] + \sin\left[\frac{\pi}{2} - (5\theta - 2\pi)\right]$$

$$= \cos\theta - \cos(3\theta - \pi) + \cos(5\theta - 2\pi)$$

$$= \cos\theta - \cos(\pi - 3\theta) + \cos(5\theta)$$ [$\cos\theta$ is even; period of 2π]

$$= \cos\theta + \cos(3\theta) + \cos(5\theta)$$

$$= \frac{\sin 6\theta}{2\sin\theta}$$ [From (i)]

$$= \frac{\sin 6\left(\frac{\pi}{2} - x\right)}{2\sin\left(\frac{\pi}{2} - x\right)}$$

$$= \frac{\sin(3\pi - 6x)}{2\cos x}$$

$$= \frac{\sin(\pi + 2\pi - 6x)}{2\cos x}$$

$$= \frac{-\sin(2\pi - 6x)}{2\cos x} = \frac{-\sin(-6x)}{2\cos x}$$ [$\sin\theta$ period 2π]

$$= \frac{\sin(6x)}{2\cos x}$$ [$\sin\theta$ is odd]

(iii) $$\int_0^{\frac{\pi}{3}} \left(\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x}\right)^2 dx = \int_0^{\frac{\pi}{3}} \left(\frac{\sin 6x}{2\cos x} \cdot \frac{2\sin x}{\sin 6x}\right)^2 dx \quad (\text{from (ii)})$$

$$= \int_0^{\frac{\pi}{3}} \tan^2\theta \, dx$$

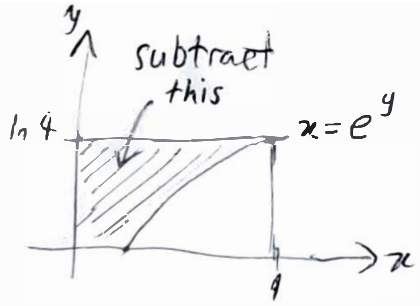
$$= \int_0^{\frac{\pi}{3}} (\sec^2 x - 1) \, dx$$

$$= \left[\tan x - x\right]_0^{\frac{\pi}{3}}$$

$$= \sqrt{3} - \frac{\pi}{3}.$$

(d) $y = \ln x$

$x = e^y$



$$V = \pi \int_0^{\ln 4} \left(4^2 - (e^y)^2\right) dy$$

$$= \pi \left[16y - \frac{e^{2y}}{2}\right]_0^{\ln 4}$$

$$= \pi\left(16\ln 4 - \frac{(e^{\ln 4})^2}{2} - 0 + \frac{1}{2}\right)$$

$$= 16\pi \ln 4 - \frac{4^2}{2}\pi + \frac{1}{2}\pi$$

$$= 16\pi \ln 4 - \frac{15\pi}{2}$$

Notes:

1) Keep this in exact form.

2) Remember to subtract the gap.

Question 14 (15 Marks) Solutions

(a) A golfer hits a golf ball from a point O with speed V m/s at an angle of 30° above the horizontal and moves freely under gravity. **4**
The ball reaches its greatest height at time T seconds after projection.

The position vector, $\underset{\sim}{r}$, at any time, t seconds, is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} \frac{\sqrt{3}}{2}Vt \\ \frac{1}{2}Vt - 5t^2 \end{pmatrix} \qquad \text{(Do } \textbf{NOT} \text{ prove)}$$

Find, in terms of V, the speed of the ball at time $\frac{2}{3}T$ seconds after projection.

$$\therefore \dot{\underset{\sim}{r}}(t) = \begin{pmatrix} \frac{\sqrt{3}}{2}V \\ \frac{1}{2}V - 10t \end{pmatrix}$$

The ball reaches its greatest height when $t = T$ and $\dot{y} = 0$ i.e. $\frac{1}{2}V - 10T = 0$

$$\therefore 10T = \frac{1}{2}V$$

$$\therefore T = \frac{1}{20}V$$

So at $t = \frac{2}{3}T = \frac{1}{30}V$:

$$\dot{\underset{\sim}{r}}\left(\frac{1}{30}V\right) = \begin{pmatrix} \frac{\sqrt{3}}{2}V \\ \frac{1}{2}V - 10 \times \frac{1}{30}V \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}V \\ \frac{1}{6}V \end{pmatrix}$$

The speed is $\left|\dot{\underset{\sim}{r}}\left(\frac{1}{30}V\right)\right|$ where $\left|\dot{\underset{\sim}{r}}\left(\frac{1}{30}V\right)\right| = \sqrt{\left(\frac{\sqrt{3}}{2}V\right)^2 + \left(\frac{1}{6}V\right)^2} = \frac{\sqrt{7}}{9}V$ m/s

Comments

Read the text carefully – do NOT prove/derive formulae given to you.

There was confusion as to speed. Some gave the velocity vector and some gave the vertical component of speed at $t = \frac{2}{3}T$.

Some students were unable to do basic fractions – so please use your calculator.

(b) Let L be the straight line passing through the point $P(x_0, y_0)$ with an angle of inclination θ to the x-axis.

The line L is defined parametrically by

$$x = x_0 + t\cos\theta$$
$$y = y_0 + t\sin\theta$$

where $t \in \mathbb{R}$ and $\theta \in [0, \pi]$.

(i) Show that for any point T on the line L that $PT = |\, t \,|$. **2**

$$\begin{aligned} PT &= \sqrt{(x - x_0)^2 + (y - y_0)^2} \\ &= \sqrt{t^2(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{t^2} \\ &= |\, t \,| \end{aligned}$$

Comment:

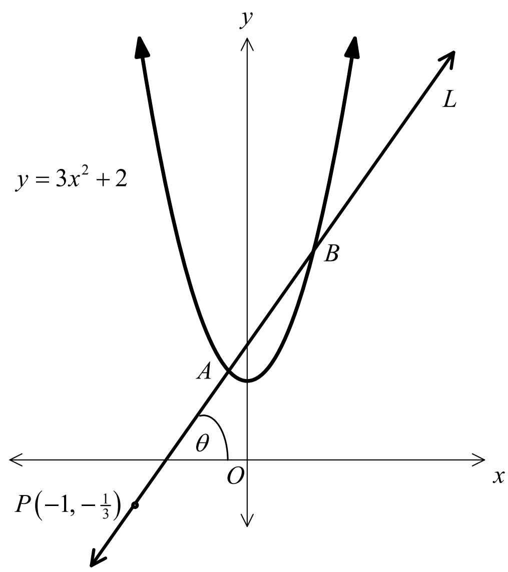
Students had to define PT to get full marks.

Too many students are not putting a subject to their algebraic statements. They are relying on the marker to decode this. This might be an eye opener to some, but the marker does not have to try and work out your logic. It should be obvious from what you write.

Some students used $(x_0, y_0) = \left(-1, -\frac{1}{3}\right)$. This was not appropriate and penalised.

This part of the question shows what the 'usefulness' of the parameter t.
It not only defines the point but it is the 'distance' of the point on the line from P.

(b) (ii) In the diagram below, L cuts the parabola $y = 3x^2 + 2$ at points A and B, where P is the point $\left(-1, -\frac{1}{3}\right)$. Let $PA = t_1$ and $PB = t_2$.



(α) Show that t_1 and t_2 are the roots of the equation **2**

$$9t^2\cos^2\theta - 3t\left(\sin\theta + 6\cos\theta\right) + 16 = 0.$$

Intersecting line L with the parabola parametrically to get points A and B:

$$y_0 + t\sin\theta = 3\left(x_0 + t\cos\theta\right)^2 + 2$$

$$x_0 = -1,\ y_0 = -\frac{1}{3}$$

The roots are $t = t_1, t_2$ i.e. the parameter for points A and B.

Note: $t_1, t_2 > 0$

$$\therefore -\frac{1}{3} + t\sin\theta = 3\left(-1 + t\cos\theta\right)^2 + 2$$

$$\therefore -1 + 3t\sin\theta = 9\left(1 - 2t\cos\theta + t^2\cos^2\theta\right) + 6$$

$$\therefore 9 - 18t\cos\theta + 9t^2\cos^2\theta + 6 + 1 - 3t\sin\theta = 0$$

$$\therefore 9t^2\cos^2\theta - 3t\left(\sin\theta + 6\cos\theta\right) + 16 = 0$$

Comment:

Deliberately or mistakenly, students assumed the result and hence they had a circular argument.

(b)	(ii)	(β)	Hence show that $AB^2 = \dfrac{\sin^2\theta + 12\sin\theta\cos\theta - 28\cos^2\theta}{9\cos^4\theta}$	**3**

$$\begin{aligned} AB^2 &= (PB - PA)^2 \\ &= (t_2 - t_1)^2 \\ &= (t_1^2 + t_2^2) - 2t_1t_2 \\ &= (t_1 + t_2)^2 - 4t_1t_2 \end{aligned}$$

$$t_1 + t_2 = \frac{3(\sin\theta + 6\cos\theta)}{9\cos^2\theta} = \frac{\sin\theta + 6\cos\theta}{3\cos^2\theta}$$

$$t_1t_2 = \frac{16}{9\cos^2\theta}$$

$$AB^2 = \left(\frac{\sin\theta + 6\cos\theta}{3\cos^2\theta}\right)^2 - 4\times\frac{16}{9\cos^2\theta}$$

$$= \frac{\sin^2\theta + 12\sin\theta\cos\theta + 36\cos^2\theta - 64\cos^2\theta}{9\cos^4\theta}$$

$$= \frac{\sin^2\theta + 12\sin\theta\cos\theta - 28\cos^2\theta}{9\cos^4\theta}$$

Comment:

Not many students did it this way.

Many just used the quadratic formula i.e. $t_2 - t_1 = \dfrac{\sqrt{\Delta}}{a}$.

Note: AB is a length i.e. $AB \neq A \times B$

(b)	(ii)	(γ)	If L is a tangent show that the two possible gradients of L are 2 and –14.	**2**

The gradient of this line is $m=\dfrac{y-y_0}{x-x_0}=\dfrac{t\sin\theta}{t\cos\theta}=\tan\theta$

L is a tangent then A and B are the same point and so $AB^2=0$.

$AB^2=0 \Leftrightarrow \sin^2\theta+12\sin\theta\cos\theta-28\cos^2\theta=0$

$\sin^2\theta+12\sin\theta\cos\theta-28\cos^2\theta=\left(\sin\theta-2\cos\theta\right)\left(\sin\theta+14\cos\theta\right)$

$AB^2=0 \Leftrightarrow \sin\theta=2\cos\theta \quad \text{or} \quad \sin\theta=-14\cos\theta$

$\therefore \tan\theta=2 \text{ or } -14$

So for L to be a tangent, the gradient must be 2 or –14.

Comment:

It was surprising to see that many students did not see the connection with (b) (ii) i.e. $\Delta=0 \Leftrightarrow AB^2=0$.

Students who wanted to use the approach of substitution i.e. $\theta=\arctan 2, \arctan(-14)$ had to show that they (both) were solutions and not just saying they were.
Many found difficulties with $\theta=\arctan(-14)$ and saying "similarly" did not work here.

An alternative solution to $\sin^2\theta+12\sin\theta\cos\theta-28\cos^2\theta=0$ was to divide both sides by $\cos^2\theta$ and get $\tan^2\theta+12\tan\theta-28=0=(\tan\theta-2)(\tan\theta+14)$.

Many students just assumed (in this case rightly) that $\tan\theta$ was the gradient of L. There were not enough marks in the question to ensure students established this.

Note: $\cos^2\theta=0$ means that $\theta=\dfrac{\pi}{2}$. If $\theta=\dfrac{\pi}{2}$ then line L is not a tangent. Hence, $\cos^2\theta\neq 0$.

(b)	(ii)	(δ)	Hence, if one of the tangents has a gradient of 2, show that $PZ = \frac{4\sqrt{5}}{3}$, where Z is the point of tangency of the line L to the parabola.	**2**

$$m = 2 = \tan\theta \Rightarrow \sin\theta = \frac{2}{\sqrt{5}}, \cos\theta = \frac{1}{\sqrt{5}}$$

Using the product of the roots with $t_1 = t_2 = t_Z\ (>0)$

$$t_Z t_Z = \frac{16}{9\cos^2\theta} \Rightarrow t_Z^2 = \frac{16}{9\cos^2\theta} = \frac{16}{9}\times 5$$

$$\therefore t_Z = \sqrt{\frac{16}{9}\times 5}$$

$$= \frac{4}{3}\sqrt{5}$$

Alternative solution

$$2t_Z = \frac{\sin\theta + 6\cos\theta}{3\cos^2\theta}$$

$$\therefore t_Z = \frac{\sin\theta + 6\cos\theta}{6\cos^2\theta}$$

$$= \frac{\frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}}}{\frac{6}{5}} = \frac{\frac{8}{\sqrt{5}}}{\frac{6}{5}}$$

$$= \frac{4}{3}\sqrt{5}$$

Comment:

This question was done well, but if the question had said "By using (b) (ii) (β), …" many students would not have fared so well. This was the intent of the question i.e. using sums and products of roots.

End of solutions